On complemented lattices.

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1. Introduction. In a recent paper the author has proved the following theorem ([5], Theorem 2) which may be considered as a converse of a well-known theorem of VON NEUMANN in the theory of complemented modular lattices:[1])

Theorem 1. *Let L be any relatively complemented lattice with greatest and least elements i, o, respectively, and let a, b, r be any elements of L such that $a \leqq r \leqq b$ holds. Let further s be any relative complement of r in $[a, b]$. If t is any solution of the equation system*

$$(1) \qquad \begin{cases} r \cap t = o, \ r \cup t = i, \\ (a \cup t) \cap b = s, \ a \cup (t \cap b) = s, \end{cases}$$

then there exists a relative complement y of a in $[o, s]$ and a relative complement z of b in $[s, i]$ such that t is a relative complement of s in $[y, z]$. Conversely, if y is any relative complement of a in $[o, s]$ and z is any relative complement of b in $[s, i]$, then any relative complement t of s in $[y, z]$ is a solution of (1).

It will be useful to complete the assertion of this theorem by the obvious

Remark. Let L, a, b be as in Theorem 1 and let r_1, r_2 be two distinct elements of $[a, b]$ which have a common relative complement s in $[a, b]$. Then, as one sees easily from Theorem 1, the two equation systems which may be obtained from (1) by substituting $r = r_1$ and $r = r_2$, respectively, have the same solutions; in particular, r_1 and r_2 have at least one common complement t.

The aim of this paper is to develop some applications of Theorem 1.

In section 2, firstly we give a condition which is necessary and sufficient for a relatively complemented lattice with greatest and least elements to

[1]) For this theorem see, e. g., [5], p. 48. — For the notations and the concepts used but not explained here, see [1].

be modular (Theorem 2). This condition is of similar kind as the condition which has been given by DILWORTH (in [2]) concerning the modularity of complemented lattices satisfying both chain conditions. Applying Theorem 2 and a generalization of the theorem of DILWORTH due to MCLAUGHLIN, we get another modularity condition for certain classes of complemented lattices (Theorem 3).

Section 3 is concerned with some contributions about the lattices with unique complements. Firstly we give a simple proof for the known theorem ([1], p. 171, ex. 2; in this paper Theorem 4) that any modular lattice with unique complements is a Boolean algebra. Afterwards, from Theorem 2 and 4, we derive the result that also any relatively complemented lattice with unique complements is distributive.

2. Modularity conditions. It is a remarkable theorem of DEDEKIND (see e. g. [1], p. 66) that a lattice L is modular if and only if no sublattice of L is isomorphic to the five-element lattice of Fig. 1. For complemented lattices satisfying the chain conditions, this theorem was sharpened by DILWORTH ([2], p. 21) in the following manner: *Let L be any complemented lattice which satisfies both the ascending chain condition and the descending one. Then L is modular if and only if no sublattice of L, including the greatest and the least elements of L, is isomorphic to the lattice of Fig. 1.*

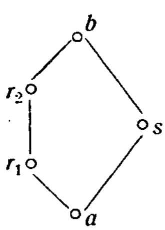

Fig. 1.

MCLAUGHLIN [4] has recently shown that the assertion of the DILWORTH theorem holds, more generally, for all atomic[2]) lattices. Now we show that it holds also for relatively complemented lattices with greatest and least elements.

For sake of brevity, we introduce the following definition: a sublattice S of a lattice with greatest element i and with least element o will be called *of Dilworth type* if $S \ni i, o$ and S isomorphic to the lattice of Fig. 1. Then we may formulate our theorem as follows:

Theorem 2. *Let L be any relatively complemented lattice with greatest and least elements. Then L is modular if and only if it contains no sublattice of Dilworth type.*

Proof. By the above-mentioned theorem of DEDEKIND, the condition is necessary. We show that it is also sufficient.

Let L be any relatively complemented lattice with greatest and least elements i, o, respectively. If L is non-modular, then — again by the theorem

[2]) A lattice L with least element o is called atomic if to each element $x(\neq o)$ of L there exists at least one element $p(\leqq x)$ which covers o.

of DEDEKIND — it contains a sublattice isomorphic to the lattice of Fig. 1. In other words, there exist elements a, b, r_1, r_2, s in L such that r_1 and r_2 have a common relative complement s in $[a, b]$. Hence, by Remark after Theorem 1, r_1 and r_2 have a common complement t. It follows that the set of elements o, r_1, r_2, t, i forms a sublattice of Dilworth type. This completes the proof of Theorem 2.

Before stating our second modularity condition announced in the introduction, we prove the following preliminary

Lemma. *Let L be any modular lattice with greatest and least elements i, o, respectively. If any elements a, b, c of L satisfy the equations*

$$a \cup c = i, \tag{2}$$

$$b \cap c = o, \tag{3}$$

then either $a \geqq b$ or a and b are incomparable.

Proof. Let V be any lattice with o and i in which the equations (2), (3) are satisfied by certain elements a, b, c such that $a < b$. It follows immediately that $a \cap c \leqq b \cap c = o$ and $b \cup c \geqq a \cup i = i$, whence

$$a \cap c = o, \quad b \cup c = i. \tag{4}$$

Next we show that for these elements also the inequalities

$$o < c < i, \quad o < a < b < i \tag{5}$$

hold. Indeed, $a < b$ holds by our assumption; further by (2) $o = c$ would imply $a = i$, by (3) $c = i$ would imply $b = o$, and both are impossible because of $a < b$; finally, by (2) $o = a$ would imply $c = i$ and by (3) $b = i$ would imply $c = o$ which we have just now shown to be impossible. It follows, by (2)—(5), that the elements o, a, b, c, i form a sublattice of V isomorphic to the lattice of Fig. 1. Hence, by the theorem of Dedekind, V is not modular.

Now we prove

Theorem 3. *Let L be any complemented lattice which has at least one of the following properties:* (i) *L is relatively complemented;* (ii) *L is atomic. Then L is modular if and only if for all elements a, b, c of L the equations* (2), (3) *and the equation $a \cap c = o$ imply that either $a \geqq b$ or a and b are incomparable.*[3])

Proof. The necessity of this condition is an obvious consequence of the Lemma. To prove its sufficiency, consider any complemented lattice L

[3]) This modularity condition is analogous te the following distributivity condition which is (implicitely) contained in the paper [3]: A complemented lattice L is distributive if the equations (2), (3) and the equation $a \cap c = o$ imply $a \geqq b$.

with the property (i) or (ii). If L is non-modular, then by our Theorem 2 or by the above-cited theorem of McLaughlin, respectively, L contains a sublattice of Dilworth type; that is, there exist elements a, b, c in L such that $a < b$ and c is a common complement of a and b. This means that, in particular, (2), (3) and $a \cap c = o$ hold for a, b, $c\,(a < b)$. Hence we conclude that if L is non-modular, then also the condition of the theorem is not satisfied. This proves the sufficiency of the condition.

We remind the reader that the assumptions (i) resp. (ii) have been used only (in the proof of the sufficiency, namely) to infer the existence of a sublattice of Dilworth type. Accordingly, they may by replaced by any assumption (iii) which assures that if a complemented non-modular lattice satisfies (iii), then it contains a sublattice of Dilworth type.

3. Theorems on lattices with unique complements. First we give a new proof for the known

Theorem 4. *Any modular lattice with unique complements is a Boolean algebra.*

Proof. By a well-known theorem ([1], p. 134., Corollary 1 of Theorem 2) it is enough to show that, in every interval of a lattice having the properties prescribed in Theorem 4, also the relative complements are uniquely determined. For this purpose let a, b, r be any elements of a complemented modular lattice L such that $a \leqq r \leqq b$. By Neumann's Theorem, L is relatively complemented; therefore, Theorem 1 may be applied for L. It follows (from the second part of Theorem 1) that to each relative complement s of r there exists (at least) one complement t of r such that $s = a \cup (t \cap b)$. Hence, if L has also the property of being a lattice with unique complements, then r has (a unique complement t and, consequently) a unique relative complement s in $[a, b]$. Thus our theorem is proved.

Finally, as a consequence of Theorems 2 and 4, we get

Theorem 5. *A lattice with unique complements is relatively complemented if and only if it is distributive.*

Proof. Since any distributive (moreover, by Neumann's Theorem, any modular) complemented lattice is relatively complemented, the "if" part of the theorem is obvious.

Conversely, let L be any lattice with unique complements. If L is non-distributive, then by Theorem 4, it is even non-modular. But it is an obvious corollary of Theorem 2 that *non-modular lattices with unique complements are not relatively complemented.* Combining these two facts, we obtain the "only if" part of Theorem 5.

References.

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